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Some weak indivisibility results in ultrahomogeneous metric spaces

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ABSTRACT

We study the validity of a partition property known as weak indivisibility for the integer and the rational Urysohn metric spaces. We also compare weak indivisibility to another partition property, called age-indivisibility, and provide an example of a countable ultrahomogeneous metric space which is age-indivisible but not weakly indivisible.

1. Introduction

The purpose of this article is the study of certain partition properties of particular metric spaces, called *ultrahomogeneous*. A metric space $\mathbf{X}$ is ultrahomogeneous when every isometry between finite metric subspaces of $\mathbf{X}$ can be extended to an isometry of $\mathbf{X}$ onto itself. For example, when seen as a metric space, any Euclidean space $\mathbb{R}^n$ has this property. So do the separable infinite dimensional Hilbert space ℓ_2 and its unit sphere $\mathbb{S}^\infty$. Another less known example of ultrahomogeneous metric space, though recently a well-studied object (see [5]), is the *Urysohn space*, denoted as $\mathbf{U}$: up to isometry, it is the unique complete separable ultrahomogeneous metric space into which every separable metric space embeds. (Here and in the rest of the paper, all the embeddings are *isometric*, that is, distance preserving.) This space also admits numerous countable analogs. For example, for various countable sets S of positive reals (see [1] for the precise condition on S), there is, up to isometry, a unique countable ultrahomogeneous metric space into which every countable metric space with distances in S embeds. When $S = \mathbb{Q}$ or $\mathbb{N}$ this gives rise to the spaces denoted respectively as $\mathbf{U}_\mathbb{Q}$ (the *rational Urysohn space*) and $\mathbf{U}_\mathbb{N}$ (the *integer Urysohn space*). Recently, separable ultrahomogeneous metric spaces have been the center of active research because of a remarkable connection between their combinatorial behavior when submitted to finite partitions and the dynamical properties of their

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isometry group. For example, consider the following result. Following the terminology used in [2], we call a metric space $\mathbf{Z} = (Z, d^{\mathbf{Z}})$ *age-indivisible* if for every finite metric subspace $\mathbf{Y}$ of $\mathbf{Z}$ and every partition $Z = B \cup R$ of the underlying set Z of $\mathbf{Z}$ (thought as a coloring of the points of Z with two colors, blue and red), the space $\mathbf{Y}$ embeds in B or R. (Here and in the whole paper, boldface characters will refer to metric structures while lightface characters will refer to the corresponding underlying sets.)

Theorem (*Folklore*)**.** *The spaces* $\mathbf{U}_{\mathbb{Q}}$ *and* $\mathbf{U}_{\mathbb{N}}$ *are age-indivisible.*

There are at least two directions for possible generalizations. First, one may ask what happens if instead of coloring the points of, say, the space $\mathbf{U}_{\mathbb{Q}}$, we color the isometric copies of a fixed finite metric subspace $\mathbf{X}$ of $\mathbf{U}_{\mathbb{Q}}$. We will not touch this subject here but Kechris, Pestov and Todorcevic showed in [3] that the answer to this question (obtained by Nešetřil in [8]) has spectacular consequences on the groups $\mathrm{iso}(\mathbf{U}_{\mathbb{Q}})$ and $\mathrm{iso}(\mathbf{U})$ of surjective self-isometries of $\mathbf{U}_{\mathbb{Q}}$ and $\mathbf{U}$. For example, every continuous action of $\mathrm{iso}(\mathbf{U})$ (equipped with the pointwise convergence topology) on a compact topological space admits a fixed point.

Another direction of generalization is to ask whether any of those spaces is *indivisible*, that is, whether B or R necessarily contains not only a copy of a fixed finite $\mathbf{Y}$ but of the whole space itself. However, it is known that any indivisible metric space must have a bounded distance set. Therefore, the spaces $\mathbf{U}_{\mathbb{Q}}$ and $\mathbf{U}_{\mathbb{N}}$ are not indivisible. Still, in this article, we investigate whether despite this obstacle, a partition result weaker than indivisibility but stronger than age-indivisibility holds. Again, following [2], we call a metric space $\mathbf{X}$ *weakly indivisible* when for every finite metric subspace $\mathbf{Y}$ of $\mathbf{X}$ and every partition $X = B \cup R$, either $\mathbf{Y}$ embeds in B or $\mathbf{X}$ embeds in R. Building on techniques developed in [6,11], we prove:

Theorem 1. *The space* $\mathbf{U}_{\mathbb{N}}$ *is weakly indivisible.*

As for $\mathbf{U}_{\mathbb{Q}}$, we are not able to prove or disprove weak indivisibility but we obtain the following weakening as a consequence of Theorem 1. If $\mathbf{X}$ is a metric space, $Y \subset X$ and $\varepsilon > 0$, $(Y)_\varepsilon$ denotes the set

$$(Y)_\varepsilon = \{x \in X : \exists y \in Y d^{\mathbf{X}}(x, y) \leq \varepsilon\}.$$

Theorem 2. *Let* $U_{\mathbb{Q}} = B \cup R$ *and* $\varepsilon > 0$*. Assume that there is a finite metric subspace* $\mathbf{Y}$ *of* $\mathbf{U}_{\mathbb{Q}}$ *that does not embed in* B*. Then* $\mathbf{U}_{\mathbb{Q}}$ *embeds in* $(R)_\varepsilon$*.*

Another consequence of Theorem 1 is the following partition result for $\mathbf{U}$.

Theorem 3. *Let* $U = B \cup R$ *and* $\varepsilon > 0$*. Assume that there is a compact metric subspace* $\mathbf{K}$ *of* $\mathbf{U}$ *that does not embed in* $(B)_\varepsilon$*. Then* $\mathbf{U}$ *embeds in* $(R)_\varepsilon$*.*

Note that these results do not answer the following: for a countable ultrahomogeneous metric space is weak indivisibility a strictly stronger property than age-indivisibility? In the last section of this paper, we indicate an example of a countable ultrahomogeneous metric space which might be age-indivisible but not weakly indivisible. To our knowledge, this could be one of the first two known examples of a countable ultrahomogeneous relational structure witnessing that weak indivisibility and age-indivisibility are distinct properties (the other example will appear in [4]). Let $\mathcal{E}_{\mathbb{Q}}$ be the class of all finite metric spaces $\mathbf{X}$ with distances in $\mathbb{Q}$ which embed isometrically into the unit sphere $\mathbb{S}^\infty$ of ℓ_2 with the property that $\{0_{\ell_2}\} \cup \mathbf{X}$ is affinely independent. It is known that there is a unique countable ultrahomogeneous metric space $\mathbb{S}^\infty_{\mathbb{Q}}$ whose class of finite metric subspaces is exactly $\mathcal{E}_{\mathbb{Q}}$, and that the metric completion of $\mathbb{S}^\infty_{\mathbb{Q}}$ is $\mathbb{S}^\infty$ (for a proof, see [9] or [10]).

Theorem 4. *The space* $\mathbb{S}^\infty_{\mathbb{Q}}$ *is age-indivisible.*

The proof of this result requires the use of a deep theorem due to Matoušek and Rödl in Euclidean Ramsey theory. As for the negation of weak indivisibility of $\mathbb{S}^\infty_{\mathbb{Q}}$, its proof is conditioned by the validity of a strong form of the Odell–Schlumprecht distortion theorem in Banach space theory, see Section 5 for more details.

The paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 3. Theorem 2 is proved in Section 4, and Theorem 4 is proved in Section 5, where a discussion of weak indivisibility of $\mathbb{S}^{\infty}_{\mathbb{Q}}$ is also included.

2. Proof of Theorem 1

The purpose of this section is to prove Theorem 1. In fact, we prove a slightly stronger result. We mentioned in introduction that there are various countable sets S of positive reals for which there is, up to isometry, a unique countable ultrahomogeneous metric space into which every countable metric space with distances in S embeds. It can be proved that when $p \in \mathbb{N}$, the integer interval $\{1, \dots, p\}$ is such a set. The corresponding countable ultrahomogeneous metric space is denoted as $\mathbf{U}_p$.

Theorem 5. *Let $U_{\mathbb{N}} = B \cup R$. Assume that there is $p \in \mathbb{N}$ such that $\mathbf{U}_p$ does not embed in B. Then $\mathbf{U}_{\mathbb{N}}$ embeds in R.*

This section is devoted to the proof of Theorem 5. We fix $p \in \mathbb{N}$ as well as a partition $U_{\mathbb{N}} = B \cup R$ such that $\mathbf{U}_p$ does not embed in B. Our goal is to prove that $\mathbf{U}_{\mathbb{N}}$ embeds into R. Let $m := \lceil p/2 \rceil$ (the least integer larger than or equal to $p/2$). Recall that if $Y \subset U_{\mathbb{N}}$, the set $(Y)_{\varepsilon}$ is defined by

$$(Y)_{\varepsilon} = \{x \in U_{\mathbb{N}} : \exists y \in Y d^{\mathbf{U}_{\mathbb{N}}}(x, y) \leq \varepsilon\}.$$

In particular, if $x \in U_{\mathbb{N}}$, the set $(\{x\})_{m-1}$ denotes the set of all elements of $U_{\mathbb{N}}$ at distance $\leq m-1$ from x. We are going to construct $\tilde{U} \subset R$ isometric to $\mathbf{U}_{\mathbb{N}}$ recursively such that for every $x \in \tilde{U}$,

$$(\{x\})_{m-1} \cap \tilde{U} \subset R.$$

More precisely, fix an enumeration $\{x_n : n \in \mathbb{N}\}$ of $U_{\mathbb{N}}$. We are going to construct $\{\tilde{x}_n : n \in \mathbb{N}\} \subset U_{\mathbb{N}}$ recursively together with a decreasing sequence $\mathbf{D}_0, \mathbf{D}_1, \dots$ of metric subspaces of $\mathbf{U}_{\mathbb{N}}$ such that $x_n \mapsto \tilde{x}_n$ is an isometry and, for every $n \in \mathbb{N}$, each $\mathbf{D}_n$ is isometric to $\mathbf{U}_{\mathbb{N}}$, $\{\tilde{x}_k : k \leq n\} \subset D_n$, and $(\{\tilde{x}_n\})_{m-1} \cap D_n \subset R$. To do so, we will need the notion of *Katětov* map as well as several technical lemmas.

Definition 1. Given a metric space $\mathbf{X} = (X, d^{\mathbf{X}})$, a map $f : X \longrightarrow (0, +\infty)$ is *Katětov over* $\mathbf{X}$ when

$$\forall x, y \in X, \quad |f(x) - f(y)| \leq d^{\mathbf{X}}(x, y) \leq f(x) + f(y).$$

Equivalently, one can extend the metric $d^{\mathbf{X}}$ to $X \cup \{f\}$ by defining, for every x, y in X, $\widehat{d^{\mathbf{X}}}(x, f) = f(x)$ and $\widehat{d^{\mathbf{X}}}(x, y) = d^{\mathbf{X}}(x, y)$. The corresponding metric space is then written $\mathbf{X} \cup \{f\}$. The set of all Katětov maps over $\mathbf{X}$ is written $E(\mathbf{X})$. For a metric subspace $\mathbf{X}$ of $\mathbf{Y}$ and a Katětov map $f \in E(\mathbf{X})$, the *orbit of f in* $\mathbf{Y}$ is the set $O(f, \mathbf{Y})$ defined by

$$O(f, \mathbf{Y}) = \{y \in Y : \forall x \in X d^{\mathbf{Y}}(y, x) = f(x)\}.$$

Any element $y \in O(f, \mathbf{Y})$ is said to *realize* f over $\mathbf{X}$. Here, the concepts of Katětov map and orbit are relevant because of the following standard reformulation of the notion of ultrahomogeneity, which will be used in the rest of the paper:

Lemma 1. *Let $\mathbf{X}$ be a countable metric space. Then $\mathbf{X}$ is ultrahomogeneous iff for every finite subspace $\mathbf{F}$ of $\mathbf{X}$ and every Katětov map f over $\mathbf{F}$, if $\mathbf{F} \cup \{f\}$ embeds into $\mathbf{X}$, then $O(f, \mathbf{X}) \neq \emptyset$.*

Proof. Postponed to Section 2.3. $\square$

Lemma 2. *Let G be a finite subset of $U_{\mathbb{N}}$, and g a Katětov map with domain G and with values in $\mathbb{N}$. Then there exists an isometric copy $\mathbf{C}$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{U}_{\mathbb{N}}$ such that:*

(i) *$G \subset C$,*
(ii) *$O(g, \mathbf{C}) \subset B$ or $O(g, \mathbf{C}) \subset R$.*

In words, Lemma 2 asserts that going to a subcopy of $\mathbf{U}_{\mathbb{N}}$ if necessary, we may assume that the orbit of g is completely included in one of the parts of the partition. Observe that as a metric space, the orbit $O(g, \mathbf{C})$ is isometric to $\mathbf{U}_n$ where $n = 2\min g$ (indeed, it is countable ultrahomogeneous with distances in $\{1, \ldots, n\}$ and embeds every countable metric space with distances in $\{1, \ldots, n\}$).

Proof. The proof of Lemma 2 can be found in [11]. More precisely, Lemma 2 can be obtained by combining Lemmas 2 and 3 [11] after having replaced $\mathbf{U}_p$ by $\mathbf{U}_{\mathbb{N}}$ in these statements. The proof of Lemma 3 [11] is elementary, while the proof of Lemma 2 [11] represents the core of [11] and is too lengthy to be presented here. These two proofs can be carried out with no modification once $\mathbf{U}_p$ has been replaced by $\mathbf{U}_{\mathbb{N}}$. $\square$

Lemma 3. *Let $G_0 \subset G$ be finite subsets of $U_{\mathbb{N}}$, and let $\mathcal{G}$ a finite family of Katětov maps with domain G and such that for all $g, g' \in \mathcal{G}$:*

$$\max(|g - g'| \restriction G_0) = \max|g - g'|,$$
$$\min((g + g') \restriction G_0) = \min(g + g'),$$
$$\min(g \restriction G_0) = \min(g).$$

Then there exists an isometric copy $\mathbf{C}$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{U}_{\mathbb{N}}$ such that:

(i) $G \cap C = G_0$,
(ii) $\forall g \in \mathcal{G}\ O(g \restriction G_0, \mathbf{C}) \subset O(g, \mathbf{U}_{\mathbb{N}})$.

Proof. Postponed to Section 2.4. $\square$

2.1. Construction of $\tilde{x}_0$ and $\mathbf{D}_0$

First, pick an arbitrary $u \in U_{\mathbb{N}}$ and consider the map $g : \{u\} \longrightarrow \mathbb{N}$ defined by $g(u) = m$. By Lemma 2, find an isometric copy $\mathbf{C}$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{U}_{\mathbb{N}}$ such that:

(i) $u \in C$,
(ii) $O(g, \mathbf{C}) \subset B$ or $O(g, \mathbf{C}) \subset R$.

Note that since g has minimum m, the orbit $O(g, \mathbf{C})$ is isometric to $\mathbf{U}_{2m}$ and therefore contains a copy of $\mathbf{U}_p$. Hence, because $\mathbf{U}_p$ does not embed in B, the inclusion $O(g, \mathbf{C}) \subset B$ is excluded and we really have $O(g, \mathbf{C}) \subset R$. Let $\tilde{x}_0 \in O(g, \mathbf{C})$ and for every $k \leq m$ let $g_k : \{u, \tilde{x}_0\} \longrightarrow \mathbb{N}$ be such that $g_k(u) = m$ and $g_k(\tilde{x}_0) = k$. The sets $G_0 = \{\tilde{x}_0\}$ and $G = \{u, \tilde{x}_0\}$, and the family $\mathcal{G} = \{g_k : k \leq m\}$ satisfy the hypotheses of Lemma 3, which allows us to obtain an isometric copy $\mathbf{D}_0$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{C}$ such that:

(i) $\{u, \tilde{x}_0\} \cap D_0 = \{\tilde{x}_0\}$,
(ii) $\forall k \leq m\ O(g_k \restriction \{\tilde{x}_0\}, \mathbf{D}_0) \subset O(g_k, \mathbf{C})$.

Note that for every $k \leq m$, we have $O(g_k, \mathbf{C}) \subset O(g, \mathbf{C}) \subset R$. Therefore, in $\mathbf{D}_0$, all the spheres around $\tilde{x}_0$ with radius $k \leq m$ are included in R. So

$$(\{\tilde{x}_0\})_{m-1} \cap D_0 \subset R.$$

2.2. Induction step

Assume that we constructed $\{\tilde{x}_k : k \leq n\} \subset U_{\mathbb{N}}$ together with a decreasing sequence $\mathbf{D}_0, \ldots, \mathbf{D}_n$ of metric subspaces of $\mathbf{U}_{\mathbb{N}}$ such that $x_k \mapsto \tilde{x}_k$ is an isometry (recall that $\{x_n : n \in \mathbb{N}\}$ is the enumeration of $U_{\mathbb{N}}$ we fixed at the beginning of the proof), each $\mathbf{D}_k$ is isometric to $\mathbf{U}_{\mathbb{N}}$, $\{\tilde{x}_k : k \leq n\} \subset D_n$ and $(\{\tilde{x}_k\})_{m-1} \cap D_n \subset R$ for every $k \leq n$. We are going to construct $\tilde{x}_{n+1}$ and $\mathbf{D}_{n+1}$. Consider the map $f : \{\tilde{x}_0, \ldots, \tilde{x}_n\} \longrightarrow \mathbb{N}$ where

$$\forall k \leq n \quad f(\tilde{x}_k) = d^{\mathbf{U}_{\mathbb{N}}}(x_k, x_{n+1}).$$

Recalling that $E(\{\tilde{x}_0, \ldots, \tilde{x}_n\})$ denotes the set of all Katětov maps from the set $\{\tilde{x}_0, \ldots, \tilde{x}_n\}$ to $\mathbb{N}$, consider the set $\mathcal{G}$ defined by

$$\{g \in E(\{\tilde{x}_0, \ldots, \tilde{x}_n\}) : \forall k \leq n(|f(\tilde{x}_k) - g(\tilde{x}_k)| \leq m - 1 \text{ and } g(\tilde{x}_k) \geq m)\}.$$

This set is finite and a repeated application of Lemma 2 allows us to construct an isometric copy $\mathbf{D}_{n+1}$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{D}_n$ such that:

(i) $\{\tilde{x}_0, \ldots, \tilde{x}_n\} \subset D_{n+1}$,
(ii) $\forall g \in \mathcal{G},\ O(g, \mathbf{D}_{n+1}) \subset B$ or R.

Note that since every $g \in \mathcal{G}$ has minimum m, the orbit $O(g, \mathbf{D}_{n+1})$ is isometric to $\mathbf{U}_{2m}$ and therefore contains a copy of $\mathbf{U}_p$. Because $\mathbf{U}_p$ does not embed in B, we consequently have

$$\forall g \in \mathcal{G}, \quad O(g, \mathbf{D}_{n+1}) \subset R.$$

Let $\tilde{x}_{n+1} \in O(f, \mathbf{D}_{n+1})$. We claim that $\tilde{x}_{n+1}$ and $\mathbf{D}_{n+1}$ are as required. Note that, because $\tilde{x}_{n+1} \in O(f, \mathbf{D}_{n+1})$, we have

$$\forall k \leq n \quad d^{\mathbf{U}_\mathbb{N}}(\tilde{x}_{n+1}, \tilde{x}_k) = f(\tilde{x}_k) = d^{\mathbf{U}_\mathbb{N}}(x_k, x_{n+1}).$$

Therefore, $x_k \mapsto \tilde{x}_k$ is an isometry. Next we prove that $(\{\tilde{x}_{n+1}\})_{m-1} \cap D_{n+1} \subset R$. Indeed, let $y \in (\{\tilde{x}_{n+1}\})_{m-1} \cap D_{n+1}$. If $d^{\mathbf{U}_\mathbb{N}}(\tilde{x}_k, y) \geq m$ for every $k \leq n$, then the map $d^{\mathbf{U}_\mathbb{N}}(\cdot, y)$ is in $\mathcal{G}$ and so $y \in O(d^{\mathbf{U}_\mathbb{N}}(\cdot, y), \mathbf{D}_{n+1}) \subset R$. Otherwise, we have $d^{\mathbf{U}_\mathbb{N}}(\tilde{x}_k, y) < m$ for some $k \leq n$ and

$$y \in (\{\tilde{x}_k\})_{m-1} \cap D_{n+1} \subset (\{\tilde{x}_k\})_{m-1} \cap D_n \subset R. \quad \square$$

2.3. *Proof of* Lemma 1

The proof is standard but we detail it here for completeness. Assume that $\mathbf{X}$ is ultrahomogeneous. Let $\varphi : \mathbf{F} \cup \{f\} \longrightarrow \mathbf{X}$ be an embedding. By ultrahomogeneity of $\mathbf{X}$, there is an isometry ψ of $\mathbf{X}$ onto itself such that:

$$\forall x \in F, \quad \psi(x) = \varphi(x).$$

Then, the point $\psi^{-1}(\varphi(f))$ is in $O(f, \mathbf{X})$.

For the converse, assume that $\{x_0, \ldots, x_n\}$ and $\{z_0, \ldots, z_n\}$ are isometric finite subspaces of $\mathbf{X}$ and that $\varphi : x_k \mapsto z_k$ is an isometry. We wish to extend φ to an isometry of $\mathbf{X}$ onto itself. We do that thanks to a back and forth method. First, extend $\{x_0, \ldots, x_n\}$ and $\{z_0, \ldots, z_n\}$ so that $\{x_k : k \in \mathbb{N}\} = \{z_k : k \in \mathbb{N}\} = X$. For $k \leq n$, let $\sigma(k) = \tau(k) = k$. Then, set $\sigma(n+1) = n+1$. Consider the map f_{n+1} defined on $\{\varphi(x_{\sigma(k)}) : k \leq n\}$ by:

$$\forall k \leq n, \quad f_{n+1}(\varphi(x_{\sigma(k)})) = d^{\mathbf{X}}(x_{\sigma(n+1)}, x_{\sigma(k)}).$$

Observe that f_{n+1} is Katětov over $\{\varphi(x_{\sigma(k)}) : k \leq n\}$ and that the space $\{\varphi(x_{\sigma(k)}) : k \leq n\} \cup \{f_{n+1}\}$ is isometric to $\{x_{\sigma(k)} : k \leq n+1\}$. By hypothesis on $\mathbf{X}$, we can consequently find a point in $O(f_{n+1}, \mathbf{X})$, call it $\varphi(x_{\sigma(n+1)})$. Next, set:

$$\tau(n+1) = \min\{k \in \mathbb{N} : z_k \notin \{\varphi(x_{\sigma(i)}) : i \leq n\}\}.$$

Consider the map g_{n+1} defined on $\{x_{\sigma(k)} : k \leq n\}$ by:

$$\forall k \leq n, \quad g_{n+1}(x_{\sigma(k)}) = d^{\mathbf{X}}(z_{\tau(n+1)}, \varphi(x_{\sigma(k)})).$$

Then g_{n+1} is Katětov over the space $\{x_{\sigma(k)} : k \leq n\}$ and the corresponding union $\{x_{\sigma(k)} : k \leq n\} \cup \{g_{n+1}\}$ is isometric to $\{\varphi(x_{\sigma(k)}) : k \leq n\} \cup \{z_{\tau(n+1)}\}$. So again, by hypothesis on $\mathbf{X}$, we can find a point in $O(g_{n+1}, \mathbf{X})$, call it $\varphi^{-1}(z_{\tau(n+1)})$. In general, if σ and τ have been defined up to m and φ has been defined on $T_m := \{x_{\sigma(0)}, \ldots, x_{\sigma(m)}\} \cup \{\varphi^{-1}(z_{\sigma(0)}), \ldots, \varphi(z_{\sigma(m)})\}$, set:

$$\sigma(m+1) = \min\{k \in \mathbb{N} : x_k \notin T_m\}.$$

Consider the map f_{m+1} defined on $\varphi(T_m)$ by:

$$\forall k \leq m, \quad \begin{cases} f_{m+1}(\varphi(x_{\sigma(k)})) = d^{\mathbf{X}}(x_{\sigma(m+1)}, x_{\sigma(k)}) \\ f_{m+1}(z_{\tau(k)}) = d^{\mathbf{X}}(x_{\sigma(m+1)}, \varphi^{-1}(z_{\tau(k)})). \end{cases}$$

Observe that f_{m+1} is Katětov over $\varphi(T_m)$ and that $\varphi(T_m \cup \{f_{m+1}\})$ is isometric to $T_m \cup \{x_{\sigma(m+1)}\}$. By hypothesis on $\mathbf{X}$, we can consequently find a point in $O(f_{m+1}, \mathbf{X})$, call it $\varphi(x_{\sigma(m+1)})$. Next, let:

$$\tau(m+1) = \min\{k \in \mathbb{N} : z_k \notin \{\varphi(x_{\sigma(i)}) : i < n+1\}\}.$$

Consider the map g_{m+1} defined on T_m by:

$$\forall k \leq m, \quad \begin{cases} g_{m+1}(x_{\sigma(k)}) = d^{\mathbf{X}}(z_{\tau(m+1)}, \varphi(x_{\sigma(k)})) \\ g_{m+1}(\varphi^{-1}(z_{\tau(k)})) = d^{\mathbf{X}}(z_{\tau(m+1)}, z_{\tau(k)}). \end{cases}$$

Then g_{m+1} is Katětov over T_m and the union $T_m \cup \{g_{m+1}\}$ is isometric to $\varphi(T_m \cup \{z_{\tau(m+1)}\})$. So again, by hypothesis on $\mathbf{X}$, we can find a point in $O(g_{m+1}, \mathbf{X})$, call it $\varphi^{-1}(z_{\tau(m+1)})$. After infinitely many steps, we are left with an isometry φ with domain $\mathbf{X} = \{x_k : k \in \mathbb{N}\}$ and range $\mathbf{X} = \{z_k : k \in \mathbb{N}\}$. This finishes the proof.

2.4. *Proof of* Lemma 3

Lemma 3 is a modified version of a result proved in [11], namely Lemma 5. Let $G_0 \subset G$ be finite subsets of $\mathbf{U}_{\mathbb{N}}$, $\mathcal{G}$ a family of Katětov maps with domain G and such that for every $g, g' \in \mathcal{G}$:

$$\max(|g - g'| \restriction G_0) = \max |g - g'|,$$
$$\min((g + g') \restriction G_0) = \min(g + g').$$

We need to produce an isometric copy $\mathbf{C}$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{U}_{\mathbb{N}}$ such that:

(i) $G \cap C = G_0$.
(ii) $\forall g \in \mathcal{G}\ O(g \restriction G_0, \mathbf{C}) \subset O(g, \mathbf{U}_{\mathbb{N}})$.

First, observe that it suffices to provide the proof assuming that G is of the form $G_0 \cup \{z\}$. The general case is then handled by repeating the procedure.

Lemma 4. *Let* $\mathbf{X}$ *be a finite subspace of* $\bigcup\{O(g \restriction G_0, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}$. *Then there is an isometry* φ *on* $\mathbf{U}_{\mathbb{N}}$ *fixing* $G_0 \cup (X \cap \bigcup\{O(g, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\})$ *and such that:*

$$\forall g \in \mathcal{G} \quad \varphi(X \cap O(g \restriction G_0, \mathbf{U}_{\mathbb{N}})) \subset O(g, \mathbf{U}_{\mathbb{N}}).$$

Proof. For $x \in X$, there is a unique element $g_x \in \mathcal{G}$ such that $x \in O(g_x \restriction G_0, \mathbf{U}_{\mathbb{N}})$. Let k be the map defined on $G_0 \cup X$ by

$$k(x) = \begin{cases} d^{\mathbf{U}_{\mathbb{N}}}(x, z) & \text{if } x \in G_0, \\ g_x(z) & \text{if } x \in X. \end{cases}$$

Claim 1. *The map* k *is Katětov.*

Proof. The metric space $G_0 \cup \{z\}$ witnesses that k is Katětov over G_0. Hence, it suffices to check that for every $x \in X$ and $y \in G_0 \cup X$,

$$|k(x) - k(y)| \leq d^{\mathbf{U}_{\mathbb{N}}}(x, y) \leq k(x) + k(y).$$

Consider first the case $y \in G_0$. Then $d^{\mathbf{U}}(x, y) = g_x(y)$ and we need to check that

$$|g_x(z) - d^{\mathbf{U}_{\mathbb{N}}}(y, z)| \leq g_x(y) \leq g_x(z) + d^{\mathbf{U}_{\mathbb{N}}}(y, z).$$

Or equivalently,

$$|g_x(z) - g_x(y)| \leq d^{\mathbf{U}_{\mathbb{N}}}(y, z) \leq g_x(z) + g_x(y).$$

But this is true since g_x is Katětov over $G_0 \cup \{z\}$. Consider now the case $y \in X$. Then $k(y) = g_y(z)$ and we need to check

$$|g_x(z) - g_y(z)| \leq d^{\mathbf{U}_{\mathbb{N}}}(x, y) \leq g_x(z) + g_y(z).$$

But since $\mathbf{X}$ is a subspace of $\bigcup\{O(g \restriction G_0, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}$, we have, for every $u \in G_0$,

$$|d^{\mathbf{U}_{\mathbb{N}}}(x, u) - d^{\mathbf{U}_{\mathbb{N}}}(u, y)| \leq d^{\mathbf{U}_{\mathbb{N}}}(x, y) \leq d^{\mathbf{U}_{\mathbb{N}}}(x, u) + d^{\mathbf{U}_{\mathbb{N}}}(x, u).$$

Since $x \in O(g_x \restriction G_0, \mathbf{U}_{\mathbb{N}})$ and $y \in O(g_y \restriction G_0, \mathbf{U}_{\mathbb{N}})$, this is equivalent to

$$|g_x(u) - g_y(u)| \leq d^{\mathbf{U}_{\mathbb{N}}}(x, y) \leq g_x(u) + g_y(u).$$

Therefore,

$$\max(|g_x - g_y| \restriction G_0) \leq d^{\mathbf{U}_{\mathbb{N}}}(x, y) \leq \min((g_x + g_y) \restriction G_0).$$

Now, by hypothesis on $\mathcal{G}$, this latter inequality remains valid if G_0 is replaced by $G_0 \cup \{z\}$. The required inequality follows. □

By ultrahomogeneity of $\mathbf{U}_{\mathbb{N}}$ (or, more precisely, by its equivalent reformulation provided in Lemma 1), we can pick a point $z' \in O(k, \mathbf{U}_{\mathbb{N}})$. The metric space $G_0 \cup (\mathbf{X} \cap \bigcup\{O(g, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}) \cup \{k\}$ being isometric to the subspace of $\mathbf{U}_{\mathbb{N}}$ supported by $G_0 \cup (X \cap \bigcup\{O(g, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}) \cup \{z\}$, so is the subspace of $\mathbf{U}_{\mathbb{N}}$ supported by $G_0 \cup (X \cap \bigcup\{O(g, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}) \cup \{z'\}$. By ultrahomogeneity again, we can find a surjective isometry φ of $\mathbf{U}_{\mathbb{N}}$ fixing $G_0 \cup (X \cap \bigcup\{O(g, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\})$ and such that $\varphi(z') = z$. Then φ is as required: let $g \in \mathcal{G}$ and $x \in O(g \restriction G_0, \mathbf{U}_{\mathbb{N}})$. Then:

$$d^{\mathbf{U}_{\mathbb{N}}}(\varphi(x), z) = d^{\mathbf{U}_{\mathbb{N}}}(\varphi(x), \varphi(z')) = d^{\mathbf{U}_{\mathbb{N}}}(x, z') = k(x) = g(z).$$

That is, $\varphi(x) \in O(g, \mathbf{U}_{\mathbb{N}})$. □

Lemma 5. *There is an isometric embedding ψ of $G_0 \cup \bigcup\{O(g \restriction G_0, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}$ into $G_0 \cup \bigcup\{O(g, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}$ fixing G_0 such that:*

$$\forall g \in \mathcal{G} \quad \psi\,(O(g \restriction G_0, \mathbf{U}_{\mathbb{N}})) \subset O(g, \mathbf{U}_{\mathbb{N}}).$$

Proof. Let $\{x_n : n \in \mathbb{N}\}$ enumerate $\bigcup\{O(g \restriction G_0, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}$. For $n \in \mathbb{N}$, let g_n be the only $g \in \mathcal{G}$ such that $x_n \in O(g_n \restriction G_0, \mathbf{U}_{\mathbb{N}})$. Apply Lemma 4 inductively to construct a sequence $(\psi_n)_{n\in\mathbb{N}}$ of surjective isometries of $\mathbf{U}_{\mathbb{N}}$ such that for every $n \in \mathbb{N}$, ψ_n fixes $G_0 \cup \psi_{n-1}(\{x_k : k < n\})$ and $\psi_n(x_n) \in O(g_n, \mathbf{U}_{\mathbb{N}})$. Then ψ defined on $G_0 \cup \{x_n : n \in \mathbb{N}\}$ by $\psi \restriction G_0 = id_{G_0}$ and $\psi(x_n) = \psi_n(x_n)$ is as required. □

We now turn to the proof of Lemma 3. Let $\mathbf{Y}$ and $\mathbf{Z}$ be the metric subspaces of $\mathbf{U}_{\mathbb{N}}$ supported by $G \cup \bigcup\{O(g, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}$ and $G_0 \cup \bigcup\{O(g \restriction G_0, \mathbf{U}_{\mathbb{N}}) : g \in \mathcal{G}\}$ respectively. Let $i_0 : \mathbf{Z} \longrightarrow \mathbf{U}_{\mathbb{N}}$ be the isometric embedding provided by the identity. By Lemma 5, the space $\mathbf{Z}$ embeds isometrically into $\mathbf{Y}$ via an isometry j_0 that fixes G_0. We can therefore consider the metric space $\mathbf{W}$ obtained by gluing $\mathbf{U}_{\mathbb{N}}$ and $\mathbf{Y}$ via an identification of $\mathbf{Z} \subset \mathbf{U}_{\mathbb{N}}$ and $j_0(\mathbf{Z}) \subset \mathbf{Y}$. The space $\mathbf{W}$ is described in Fig. 1.

Formally, the space $\mathbf{W}$ can be constructed thanks to a property of countable metric spaces with distances in $\mathbb{N}$ known as *strong amalgamation*: we can find a countable metric space $\mathbf{W}$ with distances in $\mathbb{N}$ and isometric embeddings $i_1 : \mathbf{U}_{\mathbb{N}} \longrightarrow \mathbf{W}$ and $j_1 : \mathbf{Y} \longrightarrow \mathbf{W}$ such that:

- $i_1 \circ i_0 = j_1 \circ j_0$,
- $W = i_1(U_{\mathbb{N}}) \cup j_1(Y)$,
- $i_1(U_{\mathbb{N}}) \cap j_1(Y) = (i_1 \circ i_0)(Z) = (j_1 \circ j_0)(Z)$,
- for every $x \in U_{\mathbb{N}}$ and $y \in Y$:

$$\begin{aligned} d^{\mathbf{W}}(i_1(x), j_1(y)) &= \min\{d^{\mathbf{W}}(i_1(x), i_1 \circ i_0(z)) + d^{\mathbf{W}}(j_1 \circ j_0(z), j_1(y)) : z \in Z\} \\ &= \min\{d^{\mathbf{U}_{\mathbb{N}}}(x, i_0(z)) + d^{\mathbf{Y}}(j_0(z), y) : z \in Z\} \\ &= \min\{d^{\mathbf{U}_{\mathbb{N}}}(x, z) + d^{\mathbf{Y}}(j_0(z), y) : z \in Z\}. \end{aligned}$$

The crucial point here is that in $\mathbf{W}$, every $x \in i_1(U_{\mathbb{N}})$ realizing some $g \restriction G_0$ over $i_1(G_0)$ also realizes g over $j_1(G)$.

Using $\mathbf{W}$, we show how $\mathbf{C}$ can be constructed inductively: consider an enumeration $\{x_n : n \in \mathbb{N}\}$ of $i_1(U_{\mathbb{N}})$ admitting $i_1(G_0)$ as an initial segment. Assume that the points $\varphi(x_0), \ldots, \varphi(x_n)$ are constructed so that:

- the map φ is an isometry,
- $\mathrm{dom}\varphi \subset i_1(U_{\mathbb{N}})$,
- $\varphi(x_0), \ldots, \varphi(x_n) \in U_{\mathbb{N}}$,
- $\varphi(i_1(x)) = x$ whenever $x \in G_0$,
- $d^{\mathbf{U}_{\mathbb{N}}}(\varphi(x_k), z) = d^{\mathbf{W}}(x_k, j_1(z))$ whenever $z \in G$ and $k \leq n$.

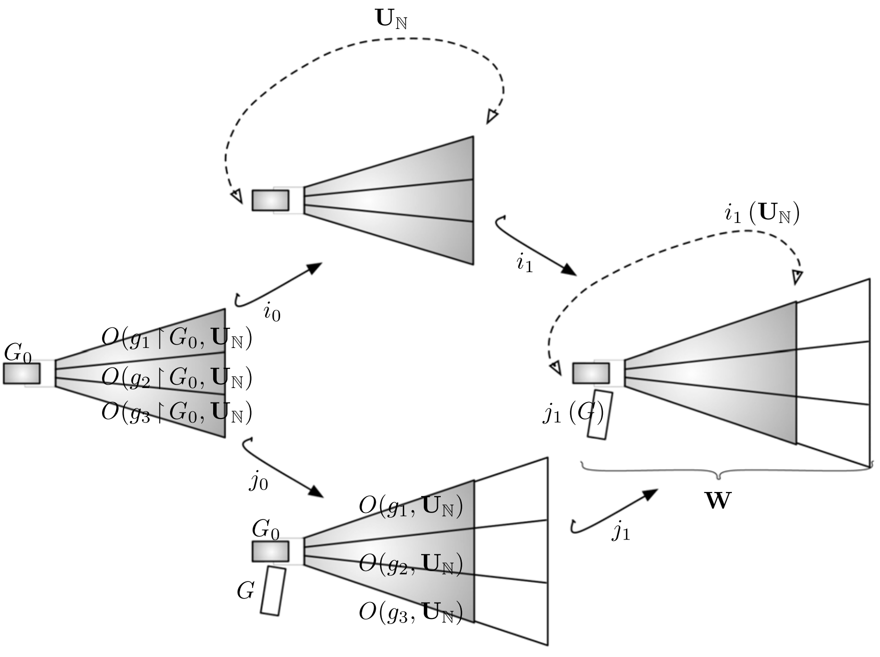

Fig. 1. The space **W**.

We want to construct $\varphi(x_{n+1})$. Consider e defined on $\{\varphi(x_k) : k \leq n\} \cup G$ by:

$$\begin{cases} \forall k \leq n \quad e(\varphi(x_k)) = d^{\mathbf{W}}(x_k, x_{n+1}), \\ \forall z \in G \quad e(z) = d^{\mathbf{W}}(j_1(z), x_{n+1}). \end{cases}$$

Observe that the metric subspace of **W** given by $\{x_k : k \leq n+1\} \cup j_1(G)$ witnesses that e is Katětov. It follows that the set $O(e, \mathbf{U}_{\mathbb{N}})$ is not empty and $\varphi(x_{n+1})$ can be chosen in it. $\square$

3. Proof of Theorem 3

The purpose of this section is to prove Theorem 3. So let $U = B \cup R$ and $\varepsilon > 0$. Assume that there is a compact metric subspace **K** of **U** that does not embed in $(B)_\varepsilon$. We wish to show that **U** embeds in $(R)_\varepsilon$.

3.1. *Proof of* Theorem 3

We will use the result of Theorem 1 as well as two technical lemmas, whose proofs are postponed to Sections 3.2 and 3.3.

Lemma 6. *Let $q \in \mathbb{N}$ be positive. Then there is an isometric copy $\mathbf{U}^*_{\mathbb{N}/q}$ of $\mathbf{U}_{\mathbb{N}/q}$ in* **U** *such that for every subspace $\widetilde{\mathbf{V}}$ of $\mathbf{U}^*_{\mathbb{N}/q}$ isometric to $\mathbf{U}_{\mathbb{N}/q}$, the set $(\tilde{V})_{1/q}$ includes an isometric copy of* **U**.

The second lemma states that in **U**, the copies of the compact space **K** can be captured by a single finite metric subspace of **U**:

Lemma 7. *There is a finite metric subspace* **Y** *of* **U** *with rational distances such that* **K** *embeds in $(\tilde{Y})_{\varepsilon/2}$ for every subspace $\widetilde{\mathbf{Y}}$ of* **U** *isometric to* **Y**.

Assuming Lemmas 6 and 7, the proof of Theorem 3 goes as follows: choose $q \in \mathbb{N}$ large enough so that $1/q \leq \varepsilon$ and all distances appearing in **Y** are integer multiples of $1/q$. The partition $U = B \cup R$ induces a partition of $U^*_{\mathbb{N}/q}$ provided by Lemma 6. Note that **Y** does not embed in B: indeed, if a subspace

$\widetilde{\mathbf{Y}}$ of B were isometric to $\mathbf{Y}$, then $(\tilde{Y})_{\varepsilon} \subset (B)_{\varepsilon}$ and by Lemma 7, the space $\mathbf{K}$ would embed in $(B)_{\varepsilon}$, which is not the case. Observe now that by weak indivisibility of the space $\mathbf{U}_{\mathbb{N}}$ (Theorem 1), the space $\mathbf{U}_{\mathbb{N}/q}$ is weakly indivisible as well, so there is a subspace $\widetilde{\mathbf{V}}$ of $\mathbf{U}^*_{\mathbb{N}/q}$ isometric to $\mathbf{U}_{\mathbb{N}/q}$ such that $\tilde{V} \subset R$. By construction of $\mathbf{U}^*_{\mathbb{N}/q}$, the set $(\tilde{V})_{1/q}$ includes an isometric copy $\widetilde{\mathbf{U}}$ of $\mathbf{U}$. To complete the proof, notice that $\tilde{U} \subset (\tilde{V})_{1/q} \subset (\tilde{V})_{\varepsilon} \subset (R)_{\varepsilon}$.

3.2. *Proof* Lemma 6

Lemma 6 is a modified version of a result proved in [6], whose statement appears at the very beginning of Proposition 5. Its proof is an easy modification of Lemma 2 [6] and is included here for completeness. The core of the proof is contained in Lemma 8 which we present now. Fix an enumeration $\{y_n : n \in \mathbb{N}\}$ of $U_{\mathbb{Q}}$. For a number α, let $\lceil \alpha \rceil_q$ denote the smallest number $\geq \alpha$ of the form l/q with l integer. The function $\lceil \cdot \rceil_q$ is subadditive and nondecreasing. Hence, the composition $\lceil d^{\mathbf{Z}} \rceil_q = \lceil \cdot \rceil_q \circ d^{\mathbf{U}_{\mathbb{Q}}}$ is a metric on $U_{\mathbb{Q}}$. Let $\mathbf{X}_q$ be the metric space $(U_{\mathbb{Q}}, \lceil d^{\mathbf{U}_{\mathbb{Q}}} \rceil_q)$. The underlying set of $\mathbf{X}_q$ is really $\{y_n : n \in \mathbb{N}\}$ but to avoid confusion, we write it $\{x_n : n \in \mathbb{N}\}$, being understood that for every $n \in \mathbb{N}$, $x_n = y_n$. On the other hand, note that $\mathbf{U}_{\mathbb{N}/q}$ and $\mathbf{X}_q$ embed isometrically into each other: $\mathbf{X}_q$ embeds in $\mathbf{U}_{\mathbb{N}/q}$ because any countable metric space with distances in $\mathbb{N}/q$ embeds in $\mathbf{U}_{\mathbb{N}/q}$, and $\mathbf{U}_{\mathbb{N}/q}$ embeds in $\mathbf{X}_q$ because any copy of $\mathbf{U}_{\mathbb{N}/q}$ in $\mathbf{U}_{\mathbb{Q}}$ remains isometric to $\mathbf{U}_{\mathbb{N}/q}$ in $\mathbf{X}_q = (U_{\mathbb{Q}}, \lceil d^{\mathbf{U}_{\mathbb{Q}}} \rceil_q)$.

Lemma 8. *There is a countable metric space* $\mathbf{Z}$ *including* $\mathbf{X}_q$ *such that for every strictly increasing* $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ *such that* $x_n \mapsto x_{\sigma(n)}$ *is an isometry, the set* $(\{x_{\sigma(n)} : n \in \mathbb{N}\})_{1/q}$ *includes an isometric copy of* $\mathbf{U}_{\mathbb{Q}}$.

Assuming Lemma 8, we now show how we can construct $\mathbf{U}^*_{\mathbb{N}/q}$. The space $\mathbf{Z}$ is countable so we may assume that it is a subspace of $\mathbf{U}$. Now, take $\mathbf{U}^*_{\mathbb{N}/q}$ a subspace of $\mathbf{X}_q$ and isometric to $\mathbf{U}_{\mathbb{N}/q}$. We claim that $\mathbf{U}^*_{\mathbb{N}/q}$ works: let $\widetilde{\mathbf{V}}$ be a subspace of $\mathbf{U}^*_{\mathbb{N}/q}$ isometric to $\mathbf{U}_{\mathbb{N}/q}$. We first show that $(\tilde{V})_{1/q}$ includes a copy of $\mathbf{U}_{\mathbb{Q}}$. The enumeration $\{x_n : n \in \mathbb{N}\}$ induces a linear ordering $<$ of $\tilde{V}$. According to Lemma 8, it suffices to show that $(\widetilde{\mathbf{V}}, <)$ includes a copy of $\{x_n : n \in \mathbb{N}\}_<$ seen as an ordered metric space. To do that, observe that since $\mathbf{X}_q$ embeds isometrically into $\mathbf{U}_{\mathbb{N}/q}$, there is a linear ordering $<^*$ of $\mathbf{U}_{\mathbb{N}/q}$ such that $\{x_n : n \in \mathbb{N}\}_<$ embeds into $(\mathbf{U}_{\mathbb{N}/q}, <^*)$ as ordered metric space. Therefore, it is enough to show:

Claim 2. $(\widetilde{\mathbf{V}}, <)$ *includes a copy of* $(\mathbf{U}_{\mathbb{N}/q}, <^*)$.

Proof. Write

$$(U_{\mathbb{N}/q}, <^*) = \{s_n : n \in \mathbb{N}\}_{<^*}$$
$$(\tilde{V}, <) = \{t_n : n \in \mathbb{N}\}_{<}.$$

Let $\sigma(0) = 0$. If $\sigma(0) < \cdots < \sigma(n)$ are chosen such that $s_k \mapsto t_{\sigma(k)}$ is a finite isometry, observe that the following set is infinite

$$\{i \in \mathbb{N} : \forall k \leqslant n \; d^{\mathbf{U}_{\mathbb{N}/q}}(t_{\sigma(k)}, t_i) = d^{\mathbf{U}_{\mathbb{N}/q}}(s_k, s_{n+1})\}.$$

Therefore, simply take $\sigma(n+1)$ in that set and larger than $\sigma(n)$. □

Observe that since the metric completion of $\mathbf{U}_{\mathbb{Q}}$ is $\mathbf{U}$, the closure of $(\tilde{V})_{1/q}$ in $\mathbf{U}$ includes a copy of $\mathbf{U}$. Hence we are done since $(\tilde{V})_{1/q}$ is closed in $\mathbf{U}$.

We now turn to the proof of Lemma 8. The strategy is first to provide the set Z where the required metric space $\mathbf{Z}$ is supposed to be based on, and then to argue that the distance $d^{\mathbf{Z}}$ can be obtained (Lemmas 9–13). To construct Z, proceed as follows: for $t \subset \mathbb{N}$, write t as the strictly increasing enumeration of its elements:

$$t = \{t_i : i \in |t|\}_<.$$

Now, let T be the set of all finite nonempty subsets t of $\mathbb{N}$ such that $x_n \mapsto x_{t_n}$ is an isometry between $\{x_n : n \in |t|\}$ and $\{x_{t_n} : n \in |t|\}$. This set T is a tree (in the order-theoretic sense) when ordered by end-extension. Let

$$Z = X_q \dot{\cup} T.$$

For $z \in Z$, define

$$\pi(z) = \begin{cases} z & \text{if } z \in X_q, \\ x_{\max z} & \text{if } z \in T. \end{cases}$$

Now, consider an edge-labelled graph structure on Z by defining δ with domain $\mathrm{dom}(\delta) \subset Z \times Z$ as follows:

- If $s, t \in T$, then $(s, t) \in \mathrm{dom}(\delta)$ iff s and t are $<_T$-comparable. In this case,
$$\delta(s, t) = d^{\mathbf{U}_{\mathbb{Q}}}(y_{|s|-1}, y_{|t|-1}).$$
- If $x, y \in X_q$, then (x, y) is always in $\mathrm{dom}(\delta)$ and
$$\delta(x, y) = d^{\mathbf{X}_q}(x, y).$$
- If $t \in T$ and $x \in X_q$, then (x, s) and (s, x) are in $\mathrm{dom}(\delta)$ iff $x = \pi(t)$. In this case
$$\delta(x, s) = \delta(s, x) = \frac{1}{q}.$$

For a branch b of T and $i \in \mathbb{N}$, let $b(i)$ be the unique element of b with height i in T. Observe that $b(i)$ is a $(i+1)$-element subset of $\mathbb{N}$. Observe also that for every $i, j \in \mathbb{N}$, $b(i)$ is connected to $\pi(b(i))$ and $b(j)$, and

(i) $\delta(b(i), \pi(b(i))) = 1/q$,
(ii) $\delta(b(i), b(j)) = d^{\mathbf{U}_{\mathbb{Q}}}(y_i, y_j)$,
(iii) $\delta(\pi(b(i)), \pi(b(j)))$ is equal to any of the following quantities:
$$d^{\mathbf{X}_q}(x_{\max b(i)}, x_{\max b(j)}) = d^{\mathbf{X}_q}(x_i, x_j) = \lceil d^{\mathbf{U}_{\mathbb{Q}}}(y_i, y_j) \rceil_q.$$

In particular, if b is a branch of T, then δ induces a metric on b and the map from $\mathbf{U}_{\mathbb{Q}}$ to b mapping y_i to $b(i)$ is a surjective isometry. We claim that if we can show that δ can be extended to a metric $d^{\mathbf{Z}}$ on Z, then Lemma 8 will be proved. Indeed, let

$$\tilde{X}_q = \{x_{\sigma(n)} : n \in \mathbb{N}\} \subset X_q,$$

with $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing and $x_n \mapsto x_{\sigma(n)}$ distance preserving. See $\mathrm{ran}(\sigma)$, the range of σ, as a branch b of T. Then $(b, d^{\mathbf{Z}}) = (b, \delta)$ is isometric to $\mathbf{U}_{\mathbb{Q}}$ and

$$b \subset (\pi(b))_{1/q} = (\tilde{X}_q)_{1/q}.$$

Our goal now is consequently to show that δ can be extended to a metric on Z with values in $[0, 1]$. For $x, y \in Z$, and $n \in \mathbb{N}$ strictly positive, define *a path from x to y of size n* as a finite sequence $\gamma = (z_i)_{i<n}$ such that $z_0 = x$, $z_{n-1} = y$ and for every $i < n-1$,

$$(z_i, z_{i+1}) \in \mathrm{dom}(\delta).$$

For x, y in Z, let $P(x, y)$ be the set of all paths from x to y. If $\gamma = (z_i)_{i<n}$ is in $P(x, y)$, $\|\gamma\|$ is defined as:

$$\|\gamma\| = \sum_{i=0}^{n-1} \delta(z_i, z_{i+1}).$$

We are going to see that the required metric can be obtained with $d^{\mathbf{Z}}$ defined by

$$d^{\mathbf{Z}}(x, y) = \inf\{\|\gamma\| : \gamma \in P(x, y)\}.$$

Equivalently, we are going to show that for every $(x, y) \in \mathrm{dom}(\delta)$, every path γ from x to y is *metric*, i.e. satisfies the following inequality:

$$\delta(x, y) \leqslant \|\gamma\|. \tag{1}$$

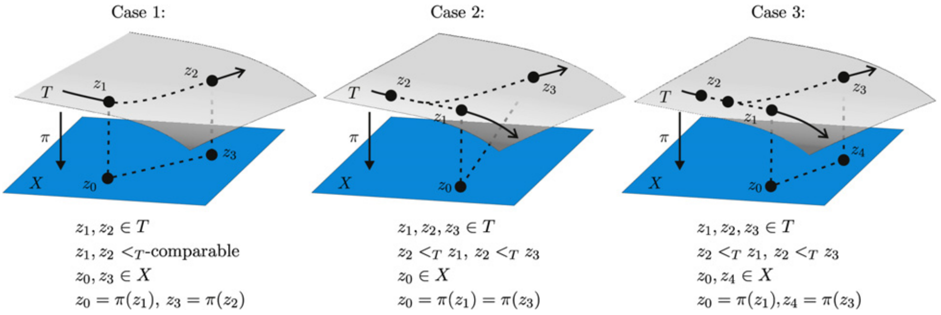

Fig. 2. Irreducible cycles.

Let $x, y \in Z$. Call a path γ from x to y *trivial* when $\gamma = (x, y)$ and *irreducible* when no proper subsequence of γ is a non-trivial path from x to y. Finally, say that γ is a *cycle* when $(x, y) \in \mathrm{dom}(\delta)$. It should be clear that to prove that $d^{\mathbf{Z}}$ works, it is enough to show that the previous inequality (1) is true for every irreducible cycle. Note that even though δ takes only rational values, it might not be the case for $d^{\mathbf{Z}}$. We now turn to the study of the irreducible cycles in Z.

Lemma 9. *Let $x, y \in T$. Assume that x and y are not $<_T$-comparable. Let γ be an irreducible path from x to y in T. Then there is $z \in T$ such that $z <_T x$, $z <_T y$ and $\gamma = (x, z, y)$.*

Proof. Write $\gamma = (z_i)_{i<n+1}$. z_1 is connected to x so z_1 is $<_T$-comparable with x. We claim that $z_1 <_T x$: otherwise, $x <_T z_1$ and every element of T which is $<_T$-comparable with z_1 is also $<_T$-comparable with x. In particular, z_2 is $<_T$-comparable with x, a contradiction since z_2 and x are not connected. We now claim that $z_1 <_T y$. Indeed, observe that $z_1 <_T z_2$: otherwise, $z_2 <_T z_1 <_T x$ so $z_2 <_T x$ contradicting irreducibility. Now, every element of T which is $<_T$-comparable with z_2 is also $<_T$-comparable with z_1, so no further element can be added to the path. Hence $z_2 = y$ and we can take $z_1 = z$. □

Lemma 10. *Every non-trivial irreducible cycle in X_q has size* 3.

Proof. Obvious since δ induces the metric $d^{\mathbf{X}_q}$ on X_q. □

Lemma 11. *Every non-trivial irreducible cycle in T has size* 3 *and is included in a branch.*

Proof. Let $c = (z_i)_{i<n}$ be a non-trivial irreducible cycle in T. We may assume that $z_0 <_T z_{n-1}$. Now, observe that every element of T which is comparable with z_0 is also comparable with z_{n-1}. In particular, z_1 is such an element. It follows that $n = 3$ and that z_0, z_1, z_2 are in a same branch. □

Lemma 12. *Every irreducible cycle in Z intersecting both X_q and T is supported by a set whose form is one of the following ones (see Fig. 2).*

Proof. Let C be a set supporting an irreducible cycle c intersecting both X_q and T. It should be clear that $|C \cap X_q| \leqslant 2$: otherwise since any two points in X_q are connected, c would admit a strict subcycle, contradicting irreducibility.

If $C \cap X_q$ has size 1, let z_0 be its unique element. In c, z_0 is connected to two elements which we denote z_1 and z_3. Note that $z_1, z_3 \in T$ so $\pi(z_1) = \pi(z_3) = z_0$. Since elements in T which are connected never project on a same point, it follows that z_1, z_3 are $<_T$-incomparable. Now, c induces an irreducible path from z_1 to z_3 in T so from Lemma 9, there is $z_2 \in C$ such that $z_2 <_T z_1$, $z_2 <_T z_3$, and we are in Case 2.

Assume now that $C \cap X_q = \{z_0, z_4\}$. Then there are $z_1, z_3 \in C \cap T$ such that $\pi(z_1) = z_0$ and $\pi(z_3) = z_4$. Note that since $z_0 \neq z_4$, we must have $z_1 \neq z_3$. Now, $C \cap T$ induces an irreducible path from z_1 to z_3 in T. By Lemma 9, either z_1 and z_3 are compatible and in this case, we are in Case 1, or z_1 and z_3 are $<_T$-incomparable and there is z_2 in $C \cap T$ such that $z_2 <_T z_1$, $z_2 <_T z_3$ and we are in Case 3. □

Lemma 13. *Every non-trivial irreducible cycle in Z is metric.*

Proof. Let c be an irreducible cycle in Z. If c is supported by X_q, then, by Lemma 10, c has size 3 and is metric since δ induces a metric on X_q. If c is supported by T, then, by Lemma 11, c also has size 3 and is included in a branch b of T. Since δ induces a metric on b, c is metric. We consequently assume that c intersects both X_q and T. According to Lemma 12, c is supported by a set C whose form is covered by one of the Cases 1, 2 or 3. So to prove the present lemma, it is enough to show every cycle obtained from a re-indexing of the cycles described in those cases is metric.

Case 1: The required inequalities are obvious after having observed that

$$\delta(z_0, z_3) = \lceil \delta(z_1, z_2) \rceil_q \quad \text{and} \quad \delta(z_0, z_1) = \delta(z_2, z_3) = \frac{1}{q}.$$

Case 2: Notice that $\delta(z_0, z_1) = \delta(z_0, z_3) = 1/q$. So the inequalities we need to prove are

$$\delta(z_1, z_2) \leqslant \delta(z_2, z_3) + \frac{2}{q}, \tag{2}$$

$$\delta(z_2, z_3) \leqslant \delta(z_1, z_2) + \frac{2}{q}. \tag{3}$$

By symmetry, it suffices to verify that (2) holds. Observe that since $\pi(z_1) = \pi(z_3) = z_0$, we must have $\lceil \delta(z_1, z_2) \rceil_q = \lceil \delta(z_2, z_3) \rceil_q$. So:

$$\delta(z_1, z_2) \leqslant \lceil \delta(z_1, z_2) \rceil_q = \lceil \delta(z_2, z_3) \rceil_q \leqslant \delta(z_2, z_3) + \frac{2}{q}.$$

Case 3: Observe that $\delta(z_0, z_1) = \delta(z_3, z_4) = 1/q$, so the inequalities we need to prove are

$$\delta(z_1, z_2) \leqslant \delta(z_2, z_3) + \delta(z_0, z_4) + \frac{2}{q}, \tag{4}$$

$$\delta(z_0, z_4) \leqslant \delta(z_1, z_2) + \delta(z_2, z_3) + \frac{2}{q}. \tag{5}$$

For (4):

$$\begin{aligned}
\delta(z_1, z_2) &\leqslant \lceil \delta(z_1, z_2) \rceil_q \\
&= \delta(\pi(z_1), \pi(z_2)) \\
&= \delta(z_0, \pi(z_2)) \\
&\leqslant \delta(z_0, z_4) + \delta(z_4, \pi(z_2)) \\
&= \delta(z_0, z_4) + \lceil \delta(z_3, z_2) \rceil_q \\
&\leqslant \delta(z_0, z_4) + \delta(z_2, z_3) + \frac{2}{q}.
\end{aligned}$$

For (5): Write $z_1 = b(j)$, $z_3 = b'(k)$, $z_2 = b(i) = b'(i)$. Then $z_0 = \pi(z_1) = x_{\max b(j)}$ and $z_4 = \pi(z_3) = x_{\max b'(k)}$. Observe also that $\delta(z_1, z_2) = d^{\mathbf{U}_\mathbb{Q}}(y_j, y_i)$ and that $\delta(z_2, z_3) = d^{\mathbf{U}_\mathbb{Q}}(y_i, y_k)$. So:

$$\begin{aligned}
\delta(z_0, z_4) &= d^{\mathbf{X}_q}(x_{\max b(j)}, x_{\max b'(k)}) \\
&\leqslant d^{\mathbf{X}_q}(x_{\max b(j)}, x_{\max b(i)}) + d^{\mathbf{X}_q}(x_{\max b'(i)}, x_{\max b'(k)}) \\
&= d^{\mathbf{X}_q}(x_j, x_i) + d^{\mathbf{X}_q}(x_i, x_k) \\
&= \left\lceil d^{\mathbf{U}_\mathbb{Q}}(y_j, y_i) \right\rceil_q + \left\lceil d^{\mathbf{U}_\mathbb{Q}}(y_i, y_k) \right\rceil_q \\
&= \lceil \delta(z_1, z_2) \rceil_q + \lceil \delta(z_2, z_3) \rceil_q \\
&\leqslant \delta(z_1, z_2) + \frac{1}{q} + \delta(z_2, z_3) + \frac{1}{q} \\
&= \delta(z_1, z_2) + \delta(z_2, z_3) + \frac{2}{q}. \quad \square
\end{aligned}$$

3.3. *Proof of* Lemma 7

Using compactness of $\mathbf{K}$, find a finite subspace $\mathbf{Z}$ of $\mathbf{K}$ such that $K \subset (Z)_{\varepsilon/2}$.

Claim 3. *The space* $\mathbf{K}$ *embeds in* $(\tilde{Z})_\varepsilon$ *for every subspace* $\widetilde{\mathbf{Z}}$ *of* $\mathbf{U}$ *isometric to* $\mathbf{Z}$.

Proof. This follows from ultrahomogeneity of $\mathbf{U}$: if $\widetilde{\mathbf{Z}}$ is a subspace of $\mathbf{U}$ isometric to $\mathbf{Z}$, let $\phi : Z \longrightarrow \tilde{Z}$ be an isometry. By ultrahomogeneity of $\mathbf{U}$, find $\Phi : U \longrightarrow U$ extending ϕ. Then $\Phi(K)$ is isometric to $\mathbf{K}$ and is included in

$$\Phi((Z)_{\varepsilon/2}) = (\Phi(Z))_{\varepsilon/2} = (\tilde{Z})_{\varepsilon/2}. \quad \square$$

Therefore, the space $\mathbf{Z}$ is almost as required except that it may not have rational distances. To arrange that, consider $q \in \mathbb{N}$ large enough so that $1/q < \varepsilon/2$. Recall that for a number α, $\lceil \alpha \rceil_q$ denotes the smallest number $\geq \alpha$ of the form l/q with l integer. The function $\lceil \cdot \rceil_q$ is subadditive and nondecreasing. Hence, the composition $\lceil d^{\mathbf{Z}} \rceil_q = \lceil \cdot \rceil_q \circ d^{\mathbf{Z}}$ is a metric on Z. Let $\mathbf{Y}$ be defined as the metric space $(Z, \lceil d^{\mathbf{Z}} \rceil_q)$. It obviously has rational distances. We are going to show that it is as required. Consider the set $X = Z \times \{0, 1\}$ and define

$$\delta((z, i), (z', i')) = \begin{cases} d^{\mathbf{Z}}(z, z') & \text{if } i = i' = 0, \\ \lceil d^{\mathbf{Z}}(z, z') \rceil_q & \text{if } i = i' = 1, \\ d^{\mathbf{Z}}(z, z') + \varepsilon/2 & \text{if } i \neq i'. \end{cases}$$

In spirit, the structure (X, δ) is obtained by putting a copy of $\mathbf{Y}(=(Z, \lceil d^{\mathbf{Z}} \rceil_q))$ above a copy of $\mathbf{Z}$ such that the distance between any point $(z, 0) \in Z \times \{0\}$ and its counterpart $(z, 1)$ in $Z \times \{1\}$ is $\varepsilon/2$.

Claim 4. *The map* δ *is a metric on* X.

Proof. The maps $d^{\mathbf{Z}}$ and $\lceil d^{\mathbf{Z}} \rceil_q$ being metrics on $Z \times \{0\}$ and $Z \times \{1\}$, it suffices to verify that the triangle inequality is satisfied on triangles of the form $\{(x, 0), (y, 0), (z, 1)\}$ and $\{(x, 1), (y, 1), (z, 0)\}$, with $x, y, z \in Z$.

Assume that $x, y, z \in Z$, and consider the triangle $\{(x, 1), (y, 1), (z, 0)\}$. Then

$$\begin{aligned} \delta((x, 1), (z, 0)) &= d^{\mathbf{Z}}(x, z) + \frac{\varepsilon}{2} \\ &\leq d^{\mathbf{Z}}(x, y) + d^{\mathbf{Z}}(y, z) + \frac{\varepsilon}{2} \\ &\leq \left\lceil d^{\mathbf{Z}}(x, y) \right\rceil_q + d^{\mathbf{Z}}(y, z) + \frac{\varepsilon}{2} \\ &\leq \delta((x, 1), (y, 1)) + \delta((y, 1), (z, 0)). \end{aligned}$$

Similarly,

$$\delta((y, 1), (z, 0)) \leq \delta((y, 1), (x, 1)) + \delta((x, 1), (z, 0)).$$

And finally,

$$\begin{aligned} \delta((x, 1), (y, 1)) &= \left\lceil d^{\mathbf{Z}}(x, y) \right\rceil_q \\ &\leq d^{\mathbf{Z}}(x, y) + \frac{1}{q} \\ &\leq d^{\mathbf{Z}}(x, y) + \frac{\varepsilon}{2} \\ &\leq d^{\mathbf{Z}}(x, z) + d^{\mathbf{Z}}(z, y) + \frac{\varepsilon}{2} \\ &\leq d^{\mathbf{Z}}(x, z) + \frac{\varepsilon}{2} + d^{\mathbf{Z}}(z, y) + \frac{\varepsilon}{2} \\ &\leq \delta((x, 1), (z, 0)) + \delta((z, 0), (y, 1)). \end{aligned}$$

Next, consider the triangle $\{(x, 0), (y, 0), (z, 1)\}$. We have

$$\begin{aligned}\delta((x,0),(z,1)) &= d^{\mathbf{Z}}(x,z)+\frac{\varepsilon}{2}\\ &\le d^{\mathbf{Z}}(x,y)+d^{\mathbf{Z}}(y,z)+\frac{\varepsilon}{2}\\ &\le \delta((x,0),(y,0))+\delta((y,0),(z,1)).\end{aligned}$$

Similarly,

$$\delta((y,0),(z,1)) \le \delta((y,0),(x,0))+\delta((x,0),(z,1)).$$

Finally,

$$\begin{aligned}\delta((x,0),(y,0)) &= d^{\mathbf{Z}}(x,y)\\ &\le d^{\mathbf{Z}}(x,z)+d^{\mathbf{Z}}(z,y)\\ &\le d^{\mathbf{Z}}(x,z)+\frac{\varepsilon}{2}+d^{\mathbf{Z}}(z,y)+\frac{\varepsilon}{2}\\ &\le \delta((x,0),(z,1))+\delta((z,1),(y,0)). \quad \square\end{aligned}$$

Denote the space (X, δ) by $\mathbf{X}$. Recall that every finite metric space embeds isometrically in $\mathbf{U}$. Hence, without loss of generality, we may suppose $Y \subset X \subset U$. We claim that $\mathbf{Y}$ is as required. By construction, the space $\mathbf{Y}$ is a finite subspace of $\mathbf{U}$ with distances in $\mathbb{N}/q$. Observe that $X \subset (Y)_{\varepsilon/2}$. Assume that a subspace $\widetilde{\mathbf{Y}}$ of $\mathbf{U}$ is isometric to $\mathbf{Y}$. By an argument similar to the one used in Claim 3, the space $\mathbf{X}$ embeds in $(\tilde{Y})_{\varepsilon/2}$. Thus, because $\mathbf{Z}$ embeds in $\mathbf{X}$, the set $(\tilde{Y})_{\varepsilon/2}$ contains a copy of $\mathbf{Z}$, call it $\widetilde{\mathbf{Z}}$. By Claim 3, the set $(\tilde{Z})_{\varepsilon/2}$ contains a copy of $\mathbf{K}$, call it $\widetilde{\mathbf{K}}$. Then

$$\tilde{K} \subset (\tilde{Z})_{\varepsilon/2} \subset ((\tilde{Y})_{\varepsilon/2})_{\varepsilon/2} \subset (\tilde{Y})_{\varepsilon}.$$

This finishes the proof of Lemma 7.

4. Proof of Theorem 2

In this section, we think of $\mathbf{U}_{\mathbb{Q}}$ as a dense metric subspace of $\mathbf{U}$. We fix a partition of $\mathbf{U}_{\mathbb{Q}}$ as well as $\varepsilon > 0$, and we assume that there is a finite metric subspace $\mathbf{Y}$ of $\mathbf{U}_{\mathbb{Q}}$ that does not embed in B. Our goal is to show that $\mathbf{U}_{\mathbb{Q}}$ embeds in $(R)_{\varepsilon}$. We start with the following technical lemma:

Lemma 14. *Let $\mathbf{V}$ be a countable subspace of $\mathbf{U}$ with rational distances. Then for every $\varepsilon > 0$ the subspace $\mathbf{U}_{\mathbb{Q}} \cap (V)_{\varepsilon}$ includes a copy of $\mathbf{V}$.*

Assuming this result for the moment, here is how we prove Theorem 2: let $U_{\mathbb{Q}} = B \cup R$ and $\varepsilon > 0$, and assume that there is a finite metric subspace $\mathbf{Y}$ of $\mathbf{U}_{\mathbb{Q}}$ that does not embed in B. We wish to show that $\mathbf{U}_{\mathbb{Q}}$ embeds in $(R)_{\varepsilon}$. Choose $q \in \mathbb{N}$ large enough so that $2/q \le \varepsilon$ and all distances appearing in $\mathbf{Y}$ are integer multiples of $1/q$. Working in $\mathbf{U}$, set

$$\begin{aligned}B^* &= \{x \in U : (\{x\})_{1/2q} \cap U_{\mathbb{Q}} \subset B\},\\ R^* &= U \smallsetminus B^* = U \cap (R)_{1/2q}.\end{aligned}$$

Consider the space $U^*_{\mathbb{N}/q}$ coming from Lemma 6. The partition $U = B^* \cup R^*$ induces a partition of $U^*_{\mathbb{N}/q}$. Observe that by weak indivisibility of $\mathbf{U}_{\mathbb{N}}$, the space $\mathbf{U}_{\mathbb{N}/q}$ is weakly indivisible as well. We also claim that the space $\mathbf{Y}$ does not embed in $U^*_{\mathbb{N}/q} \cap B^*$. Indeed, otherwise, we could find a copy $\widetilde{\mathbf{Y}}$ of $\mathbf{Y}$ in $U^*_{\mathbb{N}/q} \cap B^*$. Lemma 14 applied to $\mathbf{V} = \mathbf{Y}$ would then guarantee that $U_{\mathbb{Q}} \cap (\tilde{Y})_{1/2q}$ contains a copy of $\mathbf{Y}$. But by construction, $U_{\mathbb{Q}} \cap (\tilde{Y})_{1/2q} \subset B$. So $\mathbf{Y}$ would embed in B, a contradiction. Therefore, by weak indivisibility of $\mathbf{U}_{\mathbb{N}/q}$, there is a subspace $\mathbf{T}$ of $\mathbf{U}^*_{\mathbb{N}/q}$ isometric to $\mathbf{U}_{\mathbb{N}/q}$ such that $T \subset R^*$. By construction of $\mathbf{U}^*_{\mathbb{N}/q}$, the set $(T)_{1/q}$ includes an isometric copy of $\mathbf{U}$, hence an isometric copy $\widetilde{\mathbf{U}}$ of $\mathbf{U}_{\mathbb{Q}}$. By Lemma 14 applied to $\mathbf{V} = \widetilde{\mathbf{U}}$, the set $U_{\mathbb{Q}} \cap (\tilde{U})_{1/2q}$ contains a copy of $\mathbf{U}_{\mathbb{Q}}$. Observe now that:

$$U_{\mathbb{Q}} \cap (\tilde{U})_{1/2q} \subset ((T)_{1/q})_{1/2q} \subset ((R^*)_{1/q})_{1/2q} \subset (((R)_{1/2q})_{1/q})_{1/2q} \subset (R)_{2/q} \subset (R)_{\varepsilon}.$$

Theorem 2 is proved. The rest of this section is therefore devoted to a proof of Lemma 14.

Proof of Lemma 14. The proof of this lemma closely follows the proof of [6] Proposition 2, and is included here for completeness. We construct the required copy of $\mathbf{V}$ inductively. Consider $\{y_n : n \in \mathbb{N}\}$ an enumeration of $\mathbf{V}$. For $k \in \mathbb{N}$, set

$$\delta_k = \frac{\varepsilon}{2} \sum_{i=0}^{k} \frac{1}{2^i}.$$

Set also

$$\eta_k = \frac{\varepsilon}{3} \frac{1}{2^{k+1}}.$$

$\mathbf{U}_{\mathbb{Q}}$ being dense in $\mathbf{U}$, choose $z_0 \in \mathbf{U}_{\mathbb{Q}}$ such that $d^{\mathbf{U}}(y_0, z_0) < \delta_0$. Assume now that $z_0, \dots, z_n \in \mathbf{U}_{\mathbb{Q}}$ were constructed such that for every $k, l \leqslant n$

$$\begin{cases} d^{\mathbf{U}}(z_k, z_l) = d^{\mathbf{U}}(y_k, y_l), \\ d^{\mathbf{U}}(z_k, y_k) < \delta_k. \end{cases}$$

Again by denseness of $\mathbf{U}_{\mathbb{Q}}$ in $\mathbf{U}$, fix $z \in \mathbf{U}_{\mathbb{Q}}$ such that

$$d^{\mathbf{U}}(z, y_{n+1}) < \eta_{n+1}.$$

Then for every $k \leqslant n$,

$$\begin{aligned} \left| d^{\mathbf{U}}(z, z_k) - d^{\mathbf{U}}(y_{n+1}, y_k) \right| &= \left| d^{\mathbf{U}}(z, z_k) - d^{\mathbf{U}}(z_k, y_{n+1}) + d^{\mathbf{U}}(z_k, y_{n+1}) - d^{\mathbf{U}}(y_{n+1}, y_k) \right| \\ &\leqslant d^{\mathbf{U}}(z, y_{n+1}) + d^{\mathbf{U}}(z_k, y_k) \\ &< \eta_{n+1} + \delta_k \\ &< \eta_{n+1} + \delta_n. \end{aligned}$$

It follows that there is $z_{n+1} \in \mathbf{U}_{\mathbb{Q}}$ such that

$$\begin{cases} \forall k \leqslant n \quad d^{\mathbf{U}}(z_{n+1}, z_k) = d^{\mathbf{U}}(y_{n+1}, y_k) \\ d^{\mathbf{U}}(z_{n+1}, z) < \eta_{n+1} + \delta_n. \end{cases}$$

Indeed, consider the map f defined on $\{z_k : k \leqslant n\} \cup \{z\}$ by:

$$\begin{cases} \forall k \leqslant n \quad f(z_k) = d^{\mathbf{U}}(y_{n+1}, y_k), \\ f(z) = \max\{|d^{\mathbf{U}}(z, z_k) - d^{\mathbf{U}}(y_{n+1}, y_k)| : k \leqslant n\}. \end{cases}$$

Claim 5. *f is Katětov.*

Proof. The metric space $\{y_k : k \leqslant n+1\}$ witnesses that f is Katětov over the set $\{z_k : k \leqslant n\}$ so it suffices to prove that for every $k \leqslant n$,

$$|f(z) - f(z_k)| \leqslant d^{\mathbf{U}}(z, z_k) \leqslant f(z) + f(z_k).$$

Equivalently, for every $k \leqslant n$,

$$\left| d^{\mathbf{U}}(z, z_k) - f(z_k) \right| \leqslant f(z) \leqslant d^{\mathbf{U}}(z, z_k) + f(z_k).$$

There is nothing to do for the left-hand side because by definition of f, we have

$$f(z) = \max\{|d^{\mathbf{U}}(z, z_k) - f(z_k)| : k \leqslant n\}.$$

For right-hand side, what we need to show is that for every $k, l \leqslant n$,

$$\left| d^{\mathbf{U}}(z, z_l) - d^{\mathbf{U}}(y_{n+1}, y_l) \right| \leqslant d^{\mathbf{U}}(z, z_k) + d^{\mathbf{U}}(y_{n+1}, y_k).$$

Equivalently,

$$\begin{cases} d^{\mathbf{U}}(z, z_l) - d^{\mathbf{U}}(y_{n+1}, y_l) \leqslant d^{\mathbf{U}}(z, z_k) + d^{\mathbf{U}}(y_{n+1}, y_k), \\ d^{\mathbf{U}}(y_{n+1}, y_l) - d^{\mathbf{U}}(z, z_l) \leqslant d^{\mathbf{U}}(z, z_k) + d^{\mathbf{U}}(y_{n+1}, y_k). \end{cases}$$

The first inequality is equivalent to

$$d^{\mathbf{U}}(z, z_l) - d^{\mathbf{U}}(z, z_k) \leqslant d^{\mathbf{U}}(y_{n+1}, y_k) + d^{\mathbf{U}}(y_{n+1}, y_l).$$

But this is satisfied because

$$d^{\mathbf{U}}(z, z_l) - d^{\mathbf{U}}(z, z_k) \leqslant d^{\mathbf{U}}(z_l, z_k) = d^{\mathbf{U}}(y_k, y_l) \leqslant d^{\mathbf{U}}(y_k, y_{n+1}) + d^{\mathbf{U}}(y_{n+1}, y_l).$$

Similarly, the second inequality is equivalent to

$$d^{\mathbf{U}}(y_{n+1}, y_l) - d^{\mathbf{U}}(y_{n+1}, y_k) \leqslant d^{\mathbf{U}}(z, z_k) + d^{\mathbf{U}}(z, z_l).$$

This holds because

$$d^{\mathbf{U}}(y_{n+1}, y_l) - d^{\mathbf{U}}(y_{n+1}, y_k) \leqslant d^{\mathbf{U}}(y_k, y_l) = d^{\mathbf{U}}(z_k, z_l) \leqslant d^{\mathbf{U}}(z, z_k) + d^{\mathbf{U}}(z, z_l). \quad \square$$

The map f being Katětov, consider a point $z_{n+1} \in \mathbf{U}_{\mathbb{Q}}$ realizing f over the set $\{z_k : k \leqslant n\} \cup \{z\}$. Observe then that

$$\begin{aligned} d^{\mathbf{U}}(z_{n+1}, y_{n+1}) &\leqslant d^{\mathbf{U}}(z_{n+1}, z) + d^{\mathbf{U}}(z, y_{n+1}) \\ &< \eta_{n+1} + \delta_n + \eta_{n+1} \\ &< \delta_{n+1}. \end{aligned}$$

After infinitely many steps, we are left with $\{z_n : n \in \mathbb{N}\} \subset \mathbf{U}_{\mathbb{Q}} \cap (V)_\varepsilon$ isometric to $\mathbf{V}$. $\square$

5. Age-indivisibility may not imply weak indivisibility

In what follows, the set S is a fixed dense subset of $[0, 2]$. Let $\mathcal{E}_S$ be the class of all finite metric spaces $\mathbf{X}$ with distances in S which embed isometrically into the unit sphere $\mathbb{S}^\infty$ of ℓ_2 with the property that $\{0_{\ell_2}\} \cup \mathbf{X}$ is affinely independent.

Claim 6. *There is a unique countable ultrahomogeneous metric space $\mathbb{S}^\infty_S$ whose class of finite metric subspaces is exactly $\mathcal{E}_S$. Moreover, the metric completion of $\mathbb{S}^\infty_S$ is $\mathbb{S}^\infty$.*

Proof. See [9] or [10]. $\square$

We show:

Theorem 6. *The space $\mathbb{S}^\infty_S$ is age-indivisible.*

We also indicate why the space $\mathbb{S}^\infty_S$ may not be weakly indivisible. The proof of those results are provided in Sections 5.1 and 5.2 respectively.

5.1. *The space $\mathbb{S}^\infty_S$ is age-indivisible*

Let $\mathbf{Y}$ be a finite metric subspace of $\mathbb{S}^\infty_S$. We need to show:

Claim 7. *There is a finite metric subspace $\mathbf{Z}$ of $\mathbb{S}^\infty_S$ such that for every partition $Z = B \cup R$, the space $\mathbf{Y}$ embeds in B or R.*

The main ingredient of the proof is the following deep result due to Matoušek and Rödl:

Theorem 7 (*Matoušek–Rödl [7]*)**.** *Let $\mathbf{X}$ be an affinely independent finite metric subspace of $\mathbb{S}^\infty$ with circumradius r, and let $\alpha > 0$. Then there is a finite metric subspace $\mathbf{Z}$ of $\mathbb{S}^\infty$ with circumradius $r + \alpha$ such that for every partition $Z = B \cup R$, the space $\mathbf{X}$ embeds in B or R.*

What we need to prove is that in the case where $\mathbf{X} = \mathbf{Y}$, we may arrange $\mathbf{Z}$ to be a subspace of $\mathbb{S}^\infty_S$ (that is, with distances in S and $\{0_{\ell_2}\} \cup \mathbf{Z}$ affinely independent). We will make use of the following facts along the way:

Theorem 8 (*Schoenberg [14]*)**.** *Let $X = \{x_k : 1 \leq k \leq n\}$ be a finite set and let $\delta : X^2 \longrightarrow \mathbb{R}$ satisfying:*

(i) *for every $x \in X$, $\delta(x, x) = 0$,*
(ii) *for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$.*

Then (X, δ) is isometric to a subset of ℓ_2 iff

$$\max\left\{\sum_{1\leq i<j\leq n} \delta(x_i, x_j)^2\lambda_i\lambda_j : \sum_{k=1}^{n} \lambda_k^2 = 1 \text{ and } \sum_{k=1}^{n} \lambda_k = 0\right\} \leq 0.$$

Moreover, (X, δ) is isometric to an affinely independent subset of ℓ_2 iff the inequality is strict.

Claim 8. *Let $\mathbf{X}$ be a finite affinely independent metric subspace of $\mathbb{S}^\infty$ with circumradius r. Then there is $\varepsilon > 0$ such that for every $\delta : X^2 \longrightarrow \mathbb{R}$ satisfying*

(i) *for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$,*
(ii) $|\delta^2 - (d^{\mathbf{X}})^2| < \varepsilon^2$,

the space (X, δ) is an affinely independent metric subspace of $\mathbb{S}^\infty$.

Proof. Direct from Theorem 8 and from the fact that the map $M \mapsto Q_M$ is continuous, where for a matrix $M = (m_{ij})_{1\leq i,j\leq n}$,

$$Q_M = \max\left\{\sum_{1\leq i<j\leq n} m_{ij}\lambda_i\lambda_j : \sum_{k=1}^{n} \lambda_k^2 = 1 \text{ and } \sum_{k=1}^{n} \lambda_k = 0\right\}. \quad \square$$

Claim 9. *Let $\mathbf{X}$ be a finite metric subspace of $\mathbb{S}^\infty$ with circumradius r. Let $\varepsilon > 0$. Then $(X, \sqrt{(d^{\mathbf{X}})^2 + \varepsilon^2})$ is Euclidean, affinely independent with circumradius at most $r + \varepsilon$.*

Proof. Let V be the affine space spanned by X. Choose $(e_x)_{x\in X}$ a family of pairwise orthogonal unit vectors in $V^\perp$. For $x \in X$, set $\tilde{x} = x + \varepsilon/\sqrt{2}e_x$. Then the set $\{\tilde{x} : x \in X\}$ is affinely independent and is isometric to $(X, \sqrt{(d^{\mathbf{X}})^2 + \varepsilon^2})$. Its circumradius is at most $r + \varepsilon$ because it is contained in the ball centered at the circumcenter of X and with radius $r + \varepsilon$. $\square$

Claim 10. *Let $\mathbf{X}$ be an affinely independent subspace of $\mathbb{S}^\infty$. Then $\mathbf{X} \cup \{0_{\ell_2}\}$ is affinely independent iff the circumradius of $\mathbf{X}$ is < 1.*

Proof. Let V be the affine space spanned by X. Then the set $\mathbb{S}^\infty \cap V$ is the circumscribed sphere of X in V. It has radius <1 iff 0_{ℓ_2} does not belong to V. $\square$

Proof of Claim 7. First, we show that there is an affinely independent finite metric subspace $\mathbf{Z}_0$ of $\mathbb{S}^\infty$ with circumradius <1 such that for every partition $Z_0 = B \cup R$, $\mathbf{Y}$ embeds in B or R:

Let r denote the circumradius of $\mathbf{Y}$. Because $\mathbf{Y}$ is a subspace of $\mathbb{S}^\infty_S$, the space $\mathbf{Y} \cup \{0_{\ell_2}\}$ is affinely independent and by Claim 10, we have $r < 1$. By Claim 8, fix $\varepsilon > 0$ such that $r + 2\varepsilon < 1$ and such that for every map $\delta : Y^2 \longrightarrow \mathbb{R}$ satisfying

(i) for every $y, y' \in Y$, $\delta(y, y) = 0$ and $\delta(y', y) = \delta(y, y')$,
(ii) $|\delta^2 - (d^{\mathbf{Y}})^2| < \varepsilon^2$,

the space (Y, δ) is still Euclidean and affinely independent. Fix $\alpha > 0$ such that $\alpha < \varepsilon$. By choice of α, the space $(Y, \sqrt{(d^{\mathbf{Y}})^2 - \alpha^2})$ is still Euclidean and affinely independent. It should be clear that its circumradius is at most r. Apply Theorem 7 to produce a finite metric subspace $\mathbf{T}$ of $\mathbb{S}^\infty$ with circumradius $r + \alpha$ such that for every partition $T = B \cup R$, the space $(Y, \sqrt{(d^{\mathbf{Y}})^2 - \alpha^2})$ embeds in B or R. Set $\mathbf{Z}_0 = (T, \sqrt{(d^{\mathbf{T}})^2 + \alpha^2})$. We claim that $\mathbf{Z}_0$ is as required.

Indeed, by Claim 9, $\mathbf{Z}_0$ is Euclidean, affinely independent, and its circumradius is at most $(r+\alpha)+\alpha < r + 2\varepsilon < 1$. Next, if $Z_0 = B \cup R$, this partition induces a partition $T = B \cup R$. By construction of $\mathbf{T}$, there is a subspace $\widetilde{\mathbf{Y}}$ of $\mathbf{T}$ isometric to $(Y, \sqrt{(d^{\mathbf{Y}})^2 - \alpha^2})$ contained in B or R. Note that in $\mathbf{Z}_0$, the metric subspace supported by $\tilde{Y}$ is isometric to

$$\left(Y, \sqrt{\left(\sqrt{(d^{\mathbf{Y}})^2 - \alpha^2}\right)^2 + \alpha^2}\right) = \left(Y, \sqrt{(d^{\mathbf{Y}})^2 - \alpha^2 + \alpha^2}\right) = (Y, d^{\mathbf{Y}}) = \mathbf{Y}.$$

Consider the space $\mathbf{Z}_0$ we just constructed. Using Claim 8 as well as the denseness of S, we may modify slightly all the distances in $\mathbf{Z}_0$ that are not in S and turn $\mathbf{Z}_0$ into an affinely independent subspace $\mathbf{Z}$ of $\mathbb{S}^\infty$ with distances in S and circumradius <1. By Claim 10, the space $\{0_{\ell_2}\} \cup \mathbf{Z}$ is affinely independent. Therefore, $\mathbf{Z}$ embeds in $\mathbb{S}^\infty_S$. Finally, note that since all the distances of $\mathbf{Z}_0$ that were already in S did not get changed, the copies of $\mathbf{Y}$ in $\mathbf{Z}_0$ remain unaltered when passing to $\mathbf{Z}$. It follows that for every partition $Z = B \cup R$, the space $\mathbf{Y}$ embeds in B or R. $\square$

5.2. The space $\mathbb{S}^\infty_S$ may not be weakly indivisible

The starting point of this section is the following theorem:

Theorem 9 (*Odell–Schlumprecht [12]*). *There is a partition $\mathbb{S}^\infty = B \cup R$ and $\varepsilon > 0$ such that*

(i) *For every linear subspace V of ℓ_2 with $\dim V = \infty$, $\mathbb{S}^\infty \cap V \not\subset (B)_\varepsilon$.*
(ii) *For every linear subspace V of ℓ_2 with $\dim V = \infty$, $\mathbb{S}^\infty \cap V \not\subset (R)_\varepsilon$.*

In response to an inquiry of the authors, Thomas Schlumprecht [13] indicated that the method that was used to prove Theorem 9 in [12] (where the statement is proved first in another Banach space known as the Schlumprecht space, and then transferred to ℓ_2), can be adapted to show that $\dim V = \infty$ may be replaced by $\dim V = 2$ in (i). However, he indicated recently that some obstruction had appeared. Nevertheless, we would like to present here how the aforementioned strengthening of Theorem 9 implies that $\mathbb{S}^\infty_S$ is not weakly indivisible.

Theorem 10. *Assume that there is a partition $\mathbb{S}^\infty = B \cup R$ and $\varepsilon > 0$ such that*

(i) *for every linear subspace V of ℓ_2 with $\dim V = 2$, $\mathbb{S}^\infty \cap V \not\subset (B)_\varepsilon$,*
(ii) *for every linear subspace V of ℓ_2 with $\dim V = \infty$, $\mathbb{S}^\infty \cap V \not\subset (R)_\varepsilon$.*

Then $\mathbb{S}^\infty_S$ is not weakly indivisible.

Consider the partition of $\mathbb{S}^\infty$ provided by Theorem 10. It should be clear that it induces a partition of $\mathbb{S}^\infty_S$.

Claim 11. *$\mathbb{S}^\infty_S = B \cup R$ witnesses that $\mathbb{S}^\infty_S$ is not weakly indivisible.*

The proof makes use of the following fact, which we prove for completeness:

Claim 12. *Let $Y \subset \mathbb{S}^\infty$ be isometric to $\mathbb{S}^\infty$. Then there is a closed linear subspace V of ℓ_2 with $\dim V = \infty$ such that $Y = V \cap \mathbb{S}^\infty$.*

Proof. Consider V the closed linear span of Y in ℓ_2. Consider also the set $W = \{\lambda y : \lambda \in \mathbb{R},\ y \in Y\}$. We will be done if we show $V = W$. Clearly, $W \subset V$. For the reverse inclusion, observe that because Y is closed (it is isometric to a complete metric space), the set W is closed. Therefore, it is enough to show that all the finite linear combinations of elements of V that have norm 1 are in Y, i.e. for every $y_1, \ldots, y_n \in Y$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^n \lambda_i y_i \neq 0_{\ell_2}$,

$$\frac{\sum_{i=1}^{n} \lambda_i y_i}{\left\| \sum_{i=1}^{n} \lambda_i y_i \right\|} \in Y.$$

We proceed by induction on n. For $n = 2$, we first consider the case $\lambda_1 = \lambda_2 = 1$. Note that y_1 and y_2 cannot be antipodal (otherwise $y_1 + y_2 = 0_{\ell_2}$), and that $\frac{y_1+y_2}{\|y_1+y_2\|}$ can be characterized metrically in terms of y_1 and y_2. For example, it is the unique geodesic middle point of y_1 and y_2 in the intrinsic metric on $\mathbb{S}^\infty$. Since the intrinsic metric can be defined in terms of the usual Hilbertian metric on $\mathbb{S}^\infty$, this point must belong to Y. By a usual middle-point-type argument, it follows that the entire geodesic segment between y_1 and y_2 is contained in Y. Using then that Y is closed under antipodality (because Y being isometric to $\mathbb{S}^\infty$ any $y \in Y$ must have a point at distance 2), as well as a middle-point-type argument again, the entire great circle through y_1 and y_2 is contained in Y. That finishes the case $n = 2$. Assume that the property is proved up to $n \geq 2$. Fix $y_1, \ldots, y_{n+1} \in Y$ and $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$. Then writing

$$z = \frac{\sum_{i=1}^{n} \lambda_i y_i}{\left\| \sum_{i=1}^{n} \lambda_i y_i \right\|},$$

the vector

$$\frac{\sum_{i=1}^{n+1} \lambda_i y_i}{\left\| \sum_{i=1}^{n+1} \lambda_i y_i \right\|}$$

is a linear combination of z and y_{n+1} with norm 1. Therefore, it is of the form

$$\frac{\alpha z + \beta y_{n+1}}{\|\alpha z + \beta y_{n+1}\|}.$$

By induction hypothesis, z is in Y. So again by induction hypothesis (case $n = 2$),

$$\frac{\alpha z + \beta y_{n+1}}{\|\alpha z + \beta y_{n+1}\|} \in Y.$$

Therefore,

$$\frac{\sum_{i=1}^{n+1} \lambda_i y_i}{\left\| \sum_{i=1}^{n+1} \lambda_i y_i \right\|} \in Y. \quad \square$$

Proof of Claim 11. Let W be a linear subspace of ℓ_2 with $\dim W = 2$. By compactness of $\mathbb{S}^\infty \cap W$ and denseness of $\mathbb{S}^\infty_S$ in $\mathbb{S}^\infty$, there is $X \subset \mathbb{S}^\infty_S$ finite such that $\mathbb{S}^\infty \cap W \subset (X)_\varepsilon$. Let $\mathbf{X}$ denote the metric subspace of $\mathbb{S}^\infty_S$ supported by the set X. Then $\mathbf{X}$ does not embed in B because otherwise, there would be a linear subspace V of ℓ_2 with $\dim V = 2$ such that $\mathbb{S}^\infty \cap V \subset (B)_\varepsilon$, violating (i) of Theorem 10. On the other hand, $\mathbb{S}^\infty_S$ cannot embed in R: let $Y \subset \mathbb{S}^\infty_S$ be isometric to $\mathbb{S}^\infty_S$. Then in $\mathbb{S}^\infty$, the closure $\bar{Y}$ of Y is isometric to $\mathbb{S}^\infty$. By Claim 11, there is a closed linear subspace V of ℓ_2 with $\dim V = \infty$ such that $\bar{Y} = V \cap \mathbb{S}^\infty$. By (ii) of Theorem 10, $\bar{Y} \not\subset (R)_\varepsilon$. Therefore $\bar{Y} \not\subset R$. $\square$

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